Mutual Impedance between Probes in a Waveguide

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Abstract—**The general formulas of mutual impedance between two probes arbitrarily located in a rectangular waveguide are given by means of dyadic Green's function (DGF), field transformation, and reaction concept. The waveguide is semi-fininite. The reflection coefficient at the terminal plane ($z = 0$) is Γ. Lengths, feeding points, and orientations of the two probes in the waveguide are all arbitrary. As examples, expressions of mutual impedance for eight specific cases are given and discussed.**

I. Introduction

An important problem in the design of waveguide components is the analysis of postlike structures in a waveguide. The problem has been studied by many researchers [1]–[9]. Most investigators have been interested mainly in current distributions or equivalent impedances of posts. The mutual impedance between two probes vertically located on the broad wall of rectangular waveguide was studied recently by Ittipiboon and Shafai [10] using the vector potential $\vec{A}$ and reaction concept. The above investigations are extremely useful in designing microwave circuits, various filters, and antennas with specific uses.

In this paper, the probe field distribution and mutual coupling have been studied in detail. The general formulas of mutual impedance between probes are given. In derivation, the DGF, field transformation, and reaction theorem are used. The waveguide is semi-infinite. The reflection coefficient at the terminal plane ($z = 0$) is Γ. The lengths, feeding points, and orientations of the two probes in the waveguide are all arbitrary.

II. The Dyadic Green's Function

The problem to be considered is shown in Fig. 1. Two probe antennas, arbitrarily oriented, are located in a rectangular waveguide. Suppose the waveguide is of size $a \times b$ and is filled with air (μ_0, ϵ_0). The DGF $\bar{\bar{G}}$ of the first kind pertaining to the waveguide under study satisfies

$$\nabla \times \nabla \times \bar{\bar{G}}(\vec{r}, \vec{r}\,') - k^2 \bar{\bar{G}}(\vec{r}, \vec{r}\,') = \bar{\bar{I}}\delta(\vec{r} - \vec{r}\,') \qquad (1)$$

where k is the free-space wavenumber, $\bar{\bar{I}}$ is the unit dyadic, $r(x, y, z)$ is the field point, and $r'(x', y', z')$ is the source point. $\bar{\bar{G}}$ satisfies the boundary conditions $\hat{n} \times \bar{\bar{G}} = 0$ on the guide walls (perfect conductor) ($\hat{n}$ being the unit vector normal to guide walls), the reflection condition on the terminal plane, and the radiation condition as $z \to \infty$. The $\bar{\bar{G}}$ may be constructed by using image theory because the terminal plane ($z = 0$) with reflection coefficient Γ may be substituted by an image source [8], [9], [11]. With reference to Fig. 2, the original and the image current elements are, respectively,

$$\vec{j}(\vec{r}\,') = \frac{1}{j\omega\mu_0}(\hat{x} + \hat{y} + \hat{z})\delta(\vec{r} - \vec{r}\,') \qquad (2)$$

$$\vec{j}_i(\vec{r}_i\,') = \frac{\Gamma}{j\omega\mu_0}(\hat{x} + \hat{y} - \hat{z})\delta(\vec{r} - \vec{r}_i\,') \qquad (3)$$

where j is an imaginary number. The position vector of the image source is

$$\vec{r}_i\,' = \hat{x}x_i' + \hat{y}y_i' + \hat{z}z_i' = \hat{x}x' + \hat{y}y' + \hat{y}y' - \hat{z}z' \qquad (4)$$

The semi-infinite waveguide with terminal reflection coefficient Γ and current source $\vec{j}(\vec{r}\,')$ is equivalent to a bi-infinite waveguide without terminal plane and with sources $\vec{j}_t = \vec{j}(\vec{r}\,') + \vec{j}_i(\vec{r}_i\,')$.

The $\bar{\bar{G}}_0(\vec{r}, \vec{r}\,')$ pertaining to a bi-infinite waveguide without terminal and with source $\vec{j}(\vec{r}\,')$ is [12]–[15]

$$\bar{\bar{G}}_0(\vec{r}, \vec{r}\,') = -\frac{1}{k^2}\delta(\vec{r} - \vec{r}\,')\hat{z}\hat{z} + \frac{j}{ab}\sum_m \sum_n \frac{2 - \delta_0}{k_g k_c^2} \cdot \left[\vec{M}_{emn}(\pm k_g)\vec{M}'_{emn}(\mp k_g) + \vec{N}_{0mn}(\pm k_g)\vec{N}'_{0mn}(\mp k_g)\right], \qquad z \gtrless z' \quad (5)$$

where $k_c = [(m\pi/a)^2 + (n\pi/b)^2]^{1/2}$ and $k_g = (k^2 - k_c^2)^{1/2}$ are the eigenvalue and wavenumber of the rectangular waveguide, respectively. Kronnecker $\delta_0 = 1$ for m or $n = 0$ and 0 for m and $n \neq 0$. In the region where $z > 0$, the $\bar{\bar{G}}_{0i}(\vec{r}, \vec{r}_i\,')$ corresponding to $\vec{j}_i(\vec{r}_i\,')$ is

$$\bar{\bar{G}}_{0i}(\vec{r}, \vec{r}_i\,') = \frac{j}{ab}\sum_m \sum_n \frac{2 - \delta_0}{k_g k_c^2} \cdot \left[\vec{M}_{emn}(+k_g)\vec{M}'_{iemn}(-k_g) + \vec{N}_{0mn}(+k_g)\vec{N}'_{i0mn}(-k_g)\right. \qquad (6)$$

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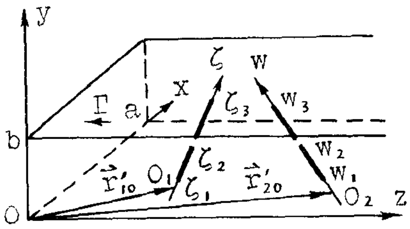

Fig. 1 Two probes in waveguide.

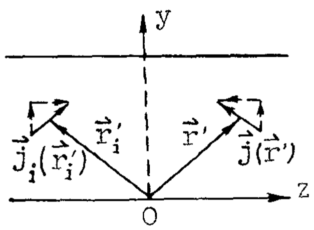

Fig. 2. The original and the image current elements.

In view of (2)–(4), $\overline{\overline{G}}_{0i}$ can be rewritten. The final expression for $\overline{\overline{G}}(\vec{r},\vec{r}')$ is given by

$$\overline{\overline{G}}3(\vec{r},\vec{r}')$$

$$= \overline{\overline{G}}_0(\vec{r},\vec{r}') + \overline{\overline{G}}_{0i}(\vec{r},\vec{r}_i')$$

$$= -\frac{1}{k^2}\delta(\vec{r}-\vec{r}')\hat{z}\hat{z} + \frac{j}{ab}\sum_m\sum_n\frac{2-\delta_0}{k_g k_c^2}$$

$$\cdot\Big[\vec{M}_{emn}(\pm k_g)\vec{M}'_{emn}(\mp k_g)$$

$$+\Gamma\vec{M}_{emn}(+k_g)\vec{M}'_{emn}(+k_g)$$

$$+\vec{N}_{0mn}(\pm k_g)\vec{N}'_{0mn}(\mp k_g)$$

$$-\Gamma\vec{N}_{0mn}(+k_g)\vec{N}'_{0mn}(+k_g)\Big]$$

$$= -\frac{1}{k^2}\delta(\vec{r}-\vec{r}')\hat{z}\hat{z} + \frac{j}{ab}\sum_m\sum_n\frac{2-\delta_0}{k_g k_c^2}$$

$$\cdot\Big[\hat{x}\hat{x}k_{km}T_{cs}T'_{cs}(e_1+\Gamma e_2) + \hat{x}\hat{y}k_{mn}T_{cs}T'_{sc}(e_1+\Gamma e_2)$$

$$+\hat{x}\hat{z}k_{gm}T_{cs}T'_{ss}(\pm e_1-\Gamma e_2) + \hat{y}\hat{x}k_{mn}T_{sc}T'_{cs}(e_1+\Gamma e_2)$$

$$+\hat{y}\hat{y}k_{kn}T_{sc}T'_{sc}(e_1+\Gamma e_2) + \hat{y}\hat{z}k_{gn}T_{sc}T'_{ss}(\pm e_1-\Gamma e_2)$$

$$+\hat{z}\hat{x}k_{gm}T_{ss}T'_{cs}(\mp e_1-\Gamma e_2) + \hat{z}\hat{y}k_{gn}T_{ss}T'_{sc}(\mp e_1-\Gamma e_2)$$

$$+\hat{z}\hat{z}k_c^2T_{ss}T'_{ss}(e_1-\Gamma e_2)\Big], \qquad z \gtrless z' \tag{7}$$

where

$$T_{cs} = \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad T'_{cs} = \cos\frac{m\pi x'}{a}\sin\frac{n\pi y'}{b}$$

$$T_{sc} = \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad T'_{sc} = \sin\frac{m\pi x'}{a}\cos\frac{n\pi y'}{b}$$

$$T_{ss} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad T'_{ss} = \sin\frac{m\pi x'}{a}\sin\frac{n\pi y'}{b}$$

$$e_1 = e^{\pm jk_g(z-z')} \qquad e_2 = e^{jk_g(z+z')}$$

$$k_{km} = k^2 - \left(\frac{m\pi}{a}\right)^2 \qquad k_{kn} = k^2 - \left(\frac{n\pi}{b}\right)^2$$

$$k_{gm} = jk_g\frac{m\pi}{a} \qquad k_{gn} = jk_g\frac{n\pi}{b}$$

$$k_{mn} = -\frac{mn\pi^2}{ab} \qquad k^2 = \omega^2\mu_0\epsilon_0.$$

III. Field $\vec{E}_1$ Radiated by Probe 1

The coordinate system of probe 1 is $O_1(\xi,\eta,\zeta)$, as shown in Fig. 1. ζ_1 and ζ_3 are the endpoints and ζ_2 is the feeding point. In the O system, the coordinates of point O_1 are (x_{10}, y_{10}, z_{10}). Assume that the current distribution of probe 1 is given by

$$\vec{J}_1(\zeta) = \hat{\zeta}I_1\delta(\xi)\delta(\eta) \tag{8a}$$

$$I_1 = \begin{cases} I_{10}\dfrac{\sin k(\zeta-\zeta_1)}{\sin k(\zeta_2-\zeta_1)}, & \zeta_1 \leq \zeta \leq \zeta_2 \\[2ex] I_{10}\dfrac{\sin k(\zeta_3-\zeta)}{\sin k(\zeta_3-\zeta_2)}, & \zeta_2 \leq \zeta \leq \zeta_3 \end{cases} \tag{8b}$$

where I_{10} is the value of the current at the feeding point. The electric field radiated from probe 1 is

$$\vec{E}_1 = j\omega\mu_0\int_{v'}\overline{\overline{G}}\cdot\vec{J}_1\,dv = j\omega\mu_0\int_{\zeta_1}^{\zeta_3}\overline{\overline{G}}\cdot\hat{\zeta}I_1\,d\zeta \tag{9}$$

Through tedious treatment of (9), the field $\vec{E}_1$ for $z > z'$ is found to be

$$\left.\begin{aligned} E_x &= \sum_m\sum_n(s_{1\cdot1}A_1T_{cs} + s_{1\cdot2}A_2T_{sc} + s_{1\cdot3}A_3T_{ss})F(z) \\ E_y &= \sum_m\sum_n(s_{2\cdot1}A_1T_{cs} + s_{2\cdot2}A_2T_{sc} + s_{2\cdot3}A_3T_{ss})F(z) \\ E_z &= \sum_m\sum_n(s_{3\cdot1}A_1T_{cs} + s_{3\cdot2}A_2T_{sc} + s_{3\cdot3}A_3T_{ss})F(z). \end{aligned}\right\} \tag{10}$$

The field distribution for $z < z'$ is

$$\left.\begin{aligned} E_x &= \sum_m\sum_n(s_{1\cdot1}B_1T_{cs} + s_{1\cdot2}B_2T_{sc} + s_{1\cdot3}B_3T_{ss})F(+\Gamma z) \\ E_y &= \sum_m\sum_n(s_{2\cdot1}B_1T_{cs} + s_{2\cdot2}B_2T_{sc} + s_{2\cdot3}B_3T_{ss})F(+\Gamma z) \\ E_z &= \sum_m\sum_n(s_{3\cdot1}B_1T_{cs} + s_{3\cdot2}B_2T_{sc} + s_{3\cdot3}B_3T_{ss})F(-\Gamma z) \end{aligned}\right\} \tag{11}$$

where

$$F(z) = e^{jk_g z} \qquad F(\pm\Gamma z) = e^{-jk_g z} \pm \Gamma e^{jk_g z}$$

$$s_{i\cdot j} = s_{j\cdot i} = \sum_{k=1}^{3} s_{ik}s_{jk}, \qquad i, j = 1,2,3$$

is the summation of corresponding term products of the ith row and the jth row in matrix (s). (s) is a transformation between O and O_1 systems. Its elements are

$$s_{11} = -\sin\psi_1\cos\theta_1 \quad s_{12} = -\cos\psi_1 \quad s_{13} = \sin\psi_1\sin\theta_1$$

$$s_{21} = \cos\psi_1\cos\theta_1 \quad s_{22} = -\sin\psi_1 \quad s_{23} = -\cos\psi_1\sin\theta_1$$

$$s_{31} = \sin\theta_1 \quad s_{32} = 0, \quad s_{33} = \cos\theta_1$$

where ψ_1, ϕ_1, and θ_1 are Eulerian angles, as shown in Fig. 3 [16], [17]. for simplification, ϕ_1 has been chosen as $\pi/2$, that is, the axis η is along the nodal line, directed to $\overline{N_1O}$. Between two coordinate systems, $(E_x, E_y, E_z)^t = (s)(E_\xi, E_\eta, E_\zeta)^t$ and $(x - x_{10}, y - y_{10}, z - z_{10})^t = (s)(\xi, \eta, \zeta)^t$, where t indicates the transpose. Parameter of (10) and (11) are as follows:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = -\frac{\eta_0}{ab}\frac{2-\delta_0}{k_g}\begin{pmatrix} k_{km} & k_{mn} & k_{gm} \\ k_{mn} & k_{kn} & k_{gn} \\ -k_{gm} & -k_{gn} & k_c^2 \end{pmatrix}\begin{pmatrix} s_{13}P_{cs} \\ s_{23}P_{sc} \\ s_{33}P_{ss} \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = -\frac{\eta_0}{ab}\frac{2-\delta_0}{k_g}\begin{pmatrix} k_{km} & k_{mn} & -k_{gm} \\ k_{mn} & k_{kn} & -k_{gn} \\ k_{gm} & k_{gn} & k_c^2 \end{pmatrix}\begin{pmatrix} s_{13}Q_{cs} \\ s_{23}Q_{sc} \\ s_{33}Q_{ss} \end{pmatrix} \tag{13}$$

$$P_{cs} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1\tau_{cs}(\zeta)f(+\Gamma\zeta)\,d\zeta$$

$$P_{sc} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1\tau_{sc}(\zeta)f(+\Gamma\zeta)\,d\zeta$$

$$P_{ss} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1\tau_{ss}(\zeta)f(-\Gamma\zeta)\,d\zeta$$

$$Q_{cs} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1\tau_{cs}(\zeta)f(\zeta)\,d\zeta$$

$$Q_{sc} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1\tau_{sc}(\zeta)f(\zeta)\,d\zeta \qquad Q_{ss} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1\tau_{ss}(\zeta)f(\zeta)\,d\zeta$$

$$\tau_{cs}(\zeta) = \cos\frac{m\pi}{a}(s_{13}\zeta + x_{10})\sin\frac{n\pi}{b}(s_{23}\zeta + y_{10})$$

$$\tau_{sc}(\zeta) = \sin\frac{m\pi}{a}(s_{13}\zeta + x_{10})\cos\frac{n\pi}{b}(s_{23}\zeta + y_{10})$$

$$\tau_{ss}(\zeta) = \sin\frac{m\pi}{a}(s_{13}\zeta + x_{10})\sin\frac{n\pi}{b}(s_{23}\zeta + y_{10})$$

$$f(\pm\Gamma\zeta) = e^{-jk_g(s_{33}\zeta + z_{10})} \pm \Gamma e^{jk_g(s_{33}\zeta + z_{10})}$$

$$f(\zeta) = e^{jk_g(s_{33}\zeta + z_{10})} \qquad \eta_0 = (\mu_0/\epsilon_0)^{1/2}.$$

By means of triangular formula, these integrals for P and Q can be readily evaluated.

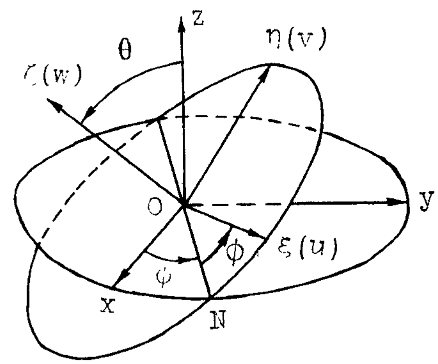

Fig 3. Eulerian angles.

IV. Mutual Impedance

In order to calculate mutual impedance, we must determine the tangent component of $\vec{E}_1$ along probe 2. With reference to Fig. 1, the coordinate system of probe 2 is $O_2(u, v, w)$. w_1 and w_3 are endpoints, and w_2 is the feeding period point. The coordinates of point O_2 in O system are (x_{20}, y_{20}, z_{20}). The current distribution of probe 2 is similar to that of probe 1(eq. (8)), that is,

$$\vec{J}_2(w) = \hat{w}I_2\delta(u)\delta(v) \tag{14a}$$

$$I_2 = \begin{cases} I_{20}\dfrac{\sin k(w - w_1)}{\sin k(w_2 - w_1)}, & w_1 \leqslant w \leqslant w_2 \\ I_{20}\dfrac{\sin k(w_3 - w)}{\sin k(w_3 - w_2)}, & w_2 \leqslant w \leqslant w_3. \end{cases} \tag{14b}$$

The transformation between the O and O_2 systems also adopts Eulerian angles $(\psi_2, \phi_2 = \pi/2, \theta_2)$. The transformation matrix is signed as (t). The elements of matrix (t) are similar to those of matrix (s), that is

$$t_{11} = -\sin\psi_2\cos\theta_2 \quad t_{12} = -\cos\psi_2 \quad t_{13} = \sin\psi_2\sin\theta_2$$

$$t_{21} = \cos\psi_2\cos\theta_2 \quad t_{22} = -\sin\psi_2 \quad t_{23} = -\cos\psi_2\sin\theta_2$$

$$t_{31} = \sin\theta_2 \quad t_{32} = 0 \quad t_{33} = \cos\theta_2.$$

The tangential component of $\vec{E}_1$ along probe 2 is given by

$$E_{1w} = \vec{E}_1\cdot\hat{w} = t_{13}E_x + t_{23}E_y + t_{33}E_z. \tag{15}$$

Because $u = v = 0$ on probe 2, the variables (x, y, z) in E_x, E_y, and E_z should be changed as follows:

$$x = t_{13}w + x_{20} \qquad y = t_{23}w + y_{20} \qquad z = t_{33}w + z_{20}.$$

Through evaluation, we get, for $z > z'$,

$$E_{1w} = \sum_m\sum_n [A_1\alpha_1(w)\tau_{cs}(w) + A_2\alpha_2(w)\tau_{sc}(w) + A_3\alpha_3(w)\tau_{ss}(w)] \tag{16}$$

whereas for $z < z'$

$$E_{1w} = \sum_m\sum_n [B_1\beta_1(w)\tau_{cs}(w) + B_2\beta_2(w)\tau_{sc}(w) + B_3\beta_3(w)\tau_{ss}(w)] \tag{17}$$

where

$$\begin{pmatrix} \alpha_1(w) \\ \alpha_2(w) \\ \alpha_3(w) \end{pmatrix} = \begin{pmatrix} s_{1\cdot 1} & s_{2\cdot 1} & s_{3\cdot 1} \\ s_{1\cdot 2} & s_{2\cdot 2} & s_{3\cdot 2} \\ s_{1\cdot 3} & s_{2\cdot 3} & s_{3\cdot 3} \end{pmatrix} \begin{pmatrix} t_{13}f(w) \\ t_{23}f(w) \\ t_{33}f(w) \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} \beta_1(w) \\ \beta_2(w) \\ \beta_3(w) \end{pmatrix} = \begin{pmatrix} s_{1\cdot 1} & s_{2\cdot 1} & s_{3\cdot 1} \\ s_{1\cdot 2} & s_{2\cdot 2} & s_{3\cdot 2} \\ s_{1\cdot 3} & s_{2\cdot 3} & s_{3\cdot 3} \end{pmatrix} \begin{pmatrix} t_{13}f(+\Gamma w) \\ t_{23}f(+\Gamma w) \\ t_{33}f(-\Gamma w) \end{pmatrix} \tag{19}$$

$$\tau_{cs}(w) = \cos\frac{m\pi}{a}(t_{13}w + x_{20})\sin\frac{n\pi}{b}(t_{23}w + y_{20})$$

$$\tau_{sc}(w) = \sin\frac{m\pi}{a}(t_{13}w + x_{20})\cos\frac{n\pi}{b}(t_{23}w + y_{20})$$

$$\tau_{ss}(w) = \sin\frac{m\pi}{a}(t_{13}w + x_{20})\sin\frac{n\pi}{b}(t_{23}w + y_{20})$$

$$f(\pm\Gamma w) = e^{-jk_g(t_{33}w + z_{20})} \pm \Gamma e^{jk_g(t_{33}w + z_{20})}$$

$$f(w) = e^{jk_g(t_{33}w + z_{20})}.$$

By the reaction concept, the mutual impedance between two probes is given by

$$M = -\frac{1}{I_{10}I_{20}}\int_{w_1}^{w_3} E_{1w}I_2\,dw \tag{20}$$

where E_{1w} is given by (16) or (17), and I_2 is given by (14). The time factor used is $e^{-j\omega t}$. If we desire to adopt $e^{+j\omega t}$, we need only replace j by $-j$ in all formulas.

V. Some Specific Cases

As examples, we discuss some useful cases. In the discussion below, suppose that the feeding points of two probes are coincident with O_1 and O_2, respectively. The heights are $h_1 = \zeta_3 - \zeta_2 (\zeta_1 = \zeta_2 = 0)$, $h_2 = w_3 - w_2 (w_1 = w_2 = 0)$, respectively.

Example A: Two Probes Perpendicular to the Same Broad Wall

Suppose two probes are parallel to the y axis, with their feeding points at $(x_{10}, 0, z_{10})$ and $(x_{20}, 0, z_{20})$, respectively, as shown in Fig. 4. In this case, $\psi_1 = \psi_2 = \pi$, $\theta_1 = \theta_2 = \pi/2$; $s_{12} = s_{23} = s_{23} = s_{31} = t_{12} = t_{23} = t_{31} = 1$; and other $s_{ij} = 0$, $t_{ij} = 0$. From (16) and (18), we get

$$E_{1w} = \sum_m \sum_n A_2\alpha_2(w)\tau_{sc}(w) = -\frac{\eta_0}{ab}\sum_m\sum_n \frac{2-\delta_0}{k_g}k_{kn}P_{sc}\tau_{sc}(w)f(w)$$

$$M = \frac{k\eta_0}{ab}\sum_m\sum_n \frac{2-\delta_0}{k_g k_{kn}} \cdot \frac{\cos kh_1 - \cos\frac{n\pi h_1}{b}}{\sin kh_1} \cdot \frac{\cos kh_2 - \cos\frac{n\pi h_2}{b}}{\sin kh_2} \cdot \sin\frac{m\pi x_{10}}{a}\sin\frac{m\pi x_{20}}{a}\left[e^{jk_g(z_{20}-z_{10})} + \Gamma e^{jk_g(z_{20}+z_{10})}\right]. \tag{21}$$

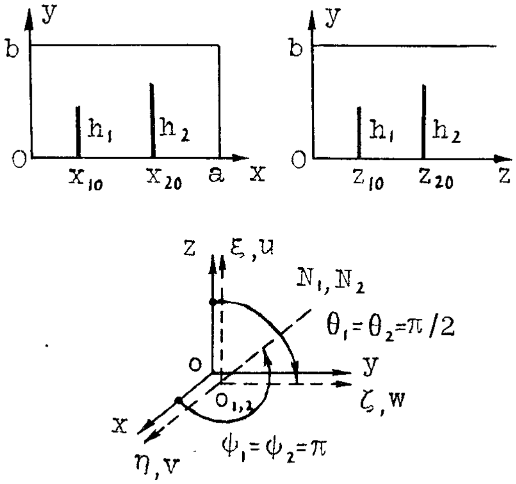

Fig. 4. Two probes on the same broad wall

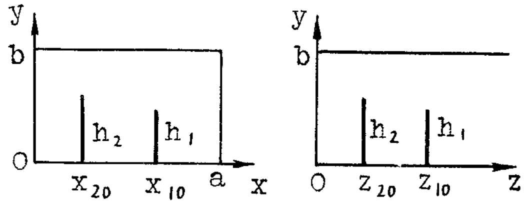

Fig. 5. Probes of Fig. 4 after position interchange.

This result is the same as [10, eq. (16)]. Because of the difference in definition of the coordinate systems, the two equations seem different in form but are identical in value. If the locations of the two probes in Fig. 4 are interchanged, as shown in Fig. 5, then (17) and (19) must used. Thus

$$E_{1w} = \sum_m\sum_n B_2\beta_2(w)\tau_{sc}(w) = -\frac{\eta_0}{ab}\sum_m\sum_n\frac{2-\delta_0}{k_g}k_{kn}Q_{sc}\tau_{sc}(w)f(+\Gamma w)$$

$$M = \sim \left[e^{jk_g(z_{10}-z_{20})} + \Gamma e^{jk_g(z_{10}+z_{20})}\right] \tag{22}$$

where the sign $\sim$ represents all terms of (21) except the terms in brackets. Comparing (21) and (22), we find that their only difference is due to the change of coordinates. Obviously, it is this result that we desire to obtain. Setting $m = 1$ and $n = 0$ in (21) and (22), we get the contribution to the mutual impedance from the dominant mode H_{10}.

Example B. Two Probes Vertical to the Opposite Broad Walls

Let us consider probes 1 and 2 parallel to the y axis with the coordinates of feed points at $(x_{10}, 0, z_{10})$ and (x_{20}, b, z_{20}), respectively, as shown in Fig. 6. For probe 1, Eulerian angles and matrix elements are as in Example A. For probe 2, $\psi_2 = 0$, $\theta_2 = \pi/2$; $t_{12} = t_{23} = -1$, $t_{31} = 1$, and the other $t_{ij} = 0$. Thus

$$M = \frac{k\eta_0}{ab}\sum_m\sum_n(-1)^{n+1}\sim \tag{23}$$

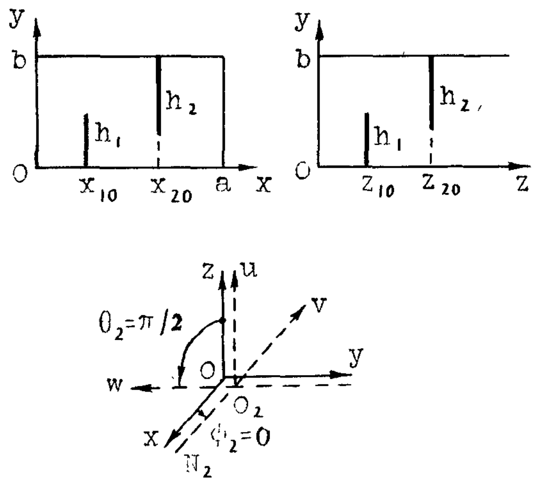

Fig. 6. Two probes on the opposite broad walls.

where the sign ~ represents all terms after double summations of (21). When n is zero or even, the direction of tangential component of the mn mode of $\vec{E}_1$ along probe 2 in Fig. 4 is the same as along probe 2 in Fig. 6. Because the current directions of probe 2 in Fig. 4 and 6 are opposite, the mutual impedances contributed by this mode have opposite signs. When n is odd, the tangential components of electric field along the two probes are opposite; hence mutual impedances are identical.

Example C. Two Probes Perpendicular to the Same Narrow Wall

Assume that the coordinates of feeding points for the two probes are $(0, y_{10}, z_{10})$ and $(0, y_{20}, z_{10})$, respectively, as shown in Fig. 7. Their Eulerian angles are $\psi_1 = \psi_2 = \theta_1 = \theta_2 = \pi/2$, the matrix elements are $s_{13} = s_{31} = t_{13} = t_{31} = 1$, $s_{22} = t_{22} = 1$, and the others are equal to zero. Thus

$$M = \frac{k\eta_0}{ab} \sum_m \sum_n \frac{2-\delta_0}{k_g k_{km}} \cdot \frac{\cos kh_1 - \cos \dfrac{m\pi h_1}{a}}{\sin kh_1} \cdot \frac{\cos kh_2 - \cos \dfrac{m\pi h_2}{a}}{\sin kh_1} \cdot \sin\frac{n\pi y_{10}}{b} \sin\frac{n\pi y_{20}}{b} \left[e^{jk_g(z_{20}-z_{10})} + \Gamma e^{jk_g(z_{20}+z_{10})} \right]. \tag{24}$$

Example D. Two Probes Vertical to the Opposite Narrow Walls

Assume that the coordinates of feeding points for the probes are $(0, y_{10}, z_{10})$ and (a, y_{20}, z_{20}), respectively, as shown in Fig. 8. For probe 1, the orientational parameters are as in Example C. For probe 2, $\psi_2 = -\pi/2$, $\theta_2 = \pi/2$; $t_{13} = -1$, $t_{22} = t_{31} = 1$, and the other $t_{ij} = 0$. Thus

$$M = \frac{k\eta_0}{ab} \sum_m \sum_n (-1)^{m+1} \sim \tag{25}$$

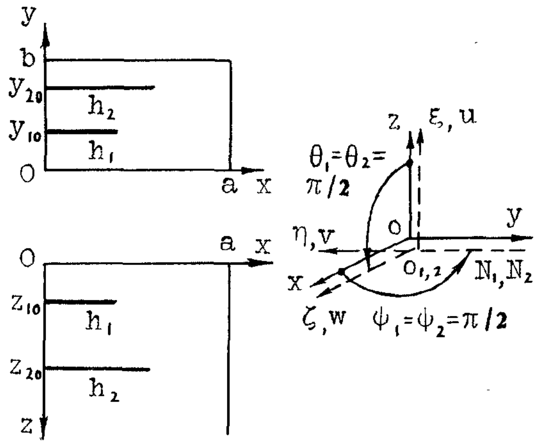

Fig. 7 Two probes on the same narrow wall.

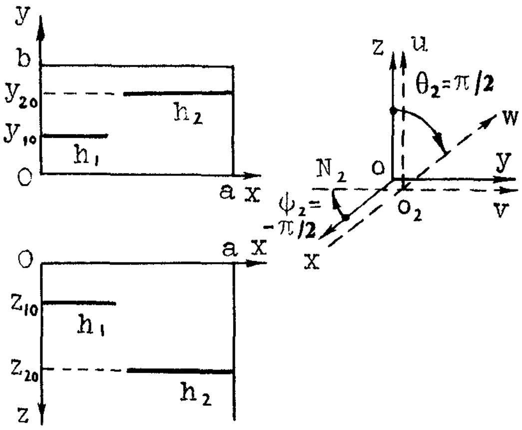

Fig. 8 Two probes on the opposite narrow walls.

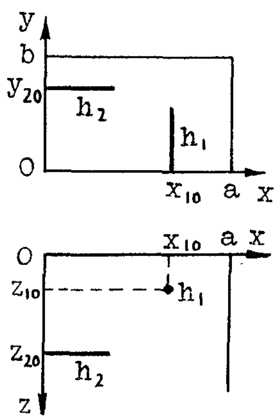

Fig. 9. One probe on the broad wall and one probe on the narrow wall.

where the sign ~ represents all terms after double summations of (24).

Example E. Two Probes Perpendicular to Broad and Narrow Walls, Respectively

Assume that the coordinates of feeding points for the two probes are $(x_{10}, 0, z_{10})$ and $(0, y_{20}, z_{20})$, respectively, as shown in Fig. 9. The Eulerian angles and matrix elements of the two probes are as in Examples A and C, respec-

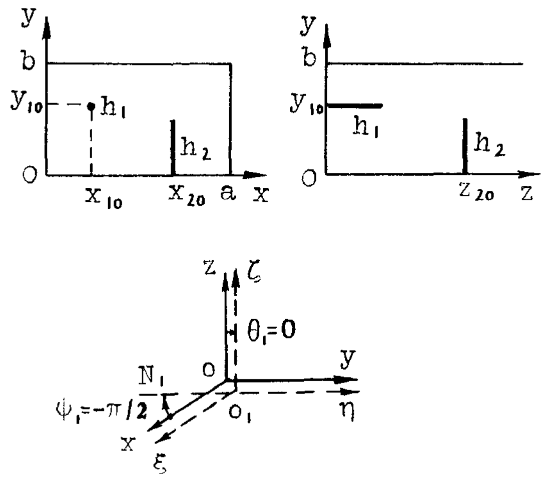

Fig. 10. One probe on the terminal wall and one probe on the broad wall.

tively. The mutual impedance is given by

$$M = \frac{k\eta_0}{ab}\sum_m\sum_n \frac{(2-\delta_0)k_{mn}}{k_g k_{km} k_{kn}} \cdot \frac{\cos kh_1 - \cos\dfrac{n\pi h_1}{b}}{\sin kh_1} \cdot \frac{\cos kh_2 - \cos\dfrac{m\pi h_2}{a}}{\sin kh_2} \cdot \sin\frac{m\pi x_{10}}{a}\sin\frac{n\pi y_{20}}{b}\left[e^{jk_g(z_{20}-z_{10})} + \Gamma e^{jk_g(z_{20}+z_{10})}\right]. \quad (26)$$

Example F. Two Probes Vertical to Terminal and Broad Walls, Respectively

For simplification, consider $\Gamma = -1$, that is, the terminal plane ($z = 0$) is constructed by a perfect conductor. Assume that the coordinates of feeding points for the two probes are $(x_{10}, y_{10}, 0)$ and $(x_{20}, 0, z_{20})$, respectively, as shown in Fig. 10. For probe 1, the orientational parameters are $\psi_1 = -\pi/2$, $\theta_1 = 0$; $s_{11} = s_{22} = s_{33} = 1$, and the other $s_{ij} = 0$. The parameters for probe 2 are as in Example A. Therefore

$$E_{1w} = -\frac{\eta_0}{ab}\sum_m\sum_n \frac{2-\delta_0}{k_g} k_{gn} P_{ss} e^{jk_g z_{20}} \sin\frac{m\pi x_{20}}{a}\cos\frac{n\pi w}{b}$$

$$M = -\frac{6k\eta_0}{ab}\sum_m\sum_n \frac{(2-\delta_0)k_{gn}}{k_g k_c^2 k_{kn}} \cdot \frac{\cos kh_1 - \cos k_g h_1}{\sin kh_1} \cdot \frac{\cos kh_2 - \cos\dfrac{n\pi h_2}{b}}{\sin kh_2} \cdot \sin\frac{m\pi x_{10}}{a}\sin\frac{n\pi y_{10}}{b}\sin\frac{m\pi x_{20}}{a}e^{jk_g z_{20}}. \quad (27)$$

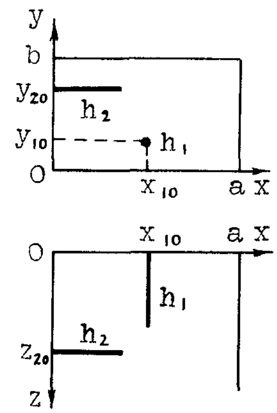

Fig. 11. One probe on the terminal wall and one probe on the narrow wall.

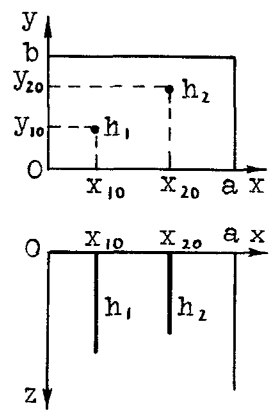

Fig. 12. Two probes on the terminal wall.

Example G. Two Probes Vertical to Terminal and Narrow Walls, Respectively

Again consider $\Gamma = -1$. The coordinates of feeding point for the two probes are $(x_{10}, y_{10}, 0)$ and $(0, y_{20}, z_{20})$, respectively, as shown in Fig. 11. The orientational parameters of probes 1 and 2 are as in Examples F and C, respectively. Hence

$$M = -\frac{6k\eta_0}{ab}\sum_m\sum_n \frac{(2-\delta_0)k_{gm}}{k_g k_c^2 k_{km}} \cdot \frac{\cos kh_1 - \cos k_g h_1}{\sin kh_1} \cdot \frac{\cos kh_2 - \cos\dfrac{m\pi h_2}{a}}{\sin kh_2} \cdot \sin\frac{m\pi x_{10}}{a}\sin\frac{n\pi y_{10}}{b}\sin\frac{n\pi y_{20}}{b}e^{jk_g z_{20}}. \quad (28)$$

Example H. Two Probes Perpendicular to Terminal Wall

Still consider $\Gamma = -1$. Assume that the coordinates of feeding points for the two probes are $(x_{10}, y_{10}, 0)$ and $(x_{20}, y_{20}, 0)$, respectively, as shown in Fig. 12. The orientational parameters of two probes are as in Example F. Thus

$$M = -\frac{1}{I_{10}I_{20}}\int_0^{h_2} E_{1w} I_2 \, dw$$

where

$$E_{1w} = \frac{6I_{10}\eta_0}{ab\sin kh_1}\sum_m\sum_n\frac{2-\delta_0}{k_g}\sin\frac{m\pi x_{10}}{a}\sin\frac{n\pi y_{10}}{b}\cdot\sin\frac{m\pi x_{20}}{a}\sin\frac{n\pi y_{20}}{b}\cdot\left\{\left[\cos kw - \cos k_g w + j\frac{k_g}{k}\sin k(h_1-w)\cos k_g w + \cos k(h_1-w)\cos k_g w\right]e^{jk_g w} - e^{jk_g h_1}\cos k_g w\right\}.$$

When $h_1 = h_2 = h$, the mutual impedance is

$$M = \frac{6k\eta_0}{ab\sin^2 kh}\sum_m\sum_n\frac{2-\delta_0}{k_g}\sin\frac{m\pi x_{10}}{a}\sin\frac{n\pi y_{10}}{b}\cdot\sin\frac{m\pi x_{20}}{a}\sin\frac{n\pi y_{20}}{b}H(k,k_g,h) \tag{29}$$

where

$$\begin{aligned}H(k,k_g,h) = {} & \frac{1}{4k^2-k_g^2}\left[(e^{jk_g h}-1)\cos kh + j\frac{2k}{k_g}\left(1-\frac{k_g^2}{2k^2}-e^{jk_g h}\right)\sin kh\right] \\ & + \frac{1}{k^2-4k_g^2}\left[-\frac{2k_g^2}{k^2}+e^{jk_g h}\cos k_g h - \left(1-\frac{2k_g^2}{k^2}\right)\cos kh + j\frac{k_g}{k}\sin kh\right] \\ & + \frac{1}{8k^2}\left[-\frac{k^2}{k^2-k_g^2}+e^{-j2k_g h}+j2k_g h + \frac{k_g^2}{k^2-k_g^2}\cos 2kh + j\frac{k_g(2k^2-k_g^2)}{k(k^2-k_g^2)}\sin 2kh\right] \\ & + \frac{1}{8k^2(k^2-k_g^2)}\left[-k_g^2(1+e^{-j2kh}) + 2k^2(e^{jk_g h}\cos k_g h + \cos 2kh)\right] \\ & + \frac{1}{k^2-k_g^2}(\cos k_g h - \cos kh)e^{jkh}.\end{aligned}$$

From (24)–(29), it is seen that the contribution to the mutual impedance from the dominant mode H_{10} is zero in Examples C–H. The higher order modes are the evanescent modes. Because these modes decay exponentially with the distance between probes, the major contribution to the mutual impedance is from the dominant mode H_{10}. However when the distance becomes small, the higher order modes may provide a large value of the mutual impedance. In over-moded guide, the mutual impedance from the higher order modes will be of interest.

VI. Conclusions

The mutual impedance between two probes in a semi-infinite rectangular waveguide has been developed. It may be seen that the mutual impedance is dependent not only on the probe lengths, orientations, and separation distance, but also the waveguide sizes, the dielectric material, and the terminal reflection coefficient. The method used in this paper is general and can be used to solve similar problems in waveguides of different cross section and also for cavity resonators.

This paper only analyzes infinitely thin probes. However, the results are also useful in the calculation of posts with definite thickness by using the wire-grid model [18].

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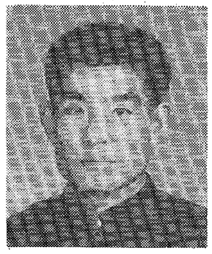

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